A METHOD FOR THE NUMERICAL SOLUTION OF A PARTICULAR SET OF COUPLED ORDINARY DIFFERENTIAL EQUATIONS

NAVAL UNDERWATER SYSTEMS CENTER
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PREFACE

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A method by which a particular set of coupled ordinary nonlinear differential equations may be solved numerically is presented. The nonlinearity is present due to the quadratic terms present in the derivatives. An example is given to illustrate the problem when applied to a model of a cable towed body moving through a viscous fluid.

TABLE OF CONTENTS

LIST OF SYMBOLS

$\vec{a}$	Acceleration vector
A	Area
A,B,C,D, and E	Square nonsingular coefficient matrices
C_1, C_2	Viscous force coefficient matrices
D	Viscous force vector
$\vec{F}$	Body force vector
G	Some general column vector
h	Time increment
H	Hydrodynamic force vector
J,K,L,M	Square nonsingular coefficient matrices
m	Mass
m_h	Hydrodynamic mass
M_s	Structural mass matrix
M_h	Hydrodynamic mass matrix
n	Direction index
q_i	Displacement in the i^{th} direction
Q	Displacement vector
S	Body surface
t	Time
T	Towline tension vector
u	Dummy variable = $\frac{dy}{dt}$
v	Dummy variable = $\frac{dz}{dt}$
φ	Normalized velocity potential

A METHOD FOR THE NUMERICAL SOLUTION OF A PARTICULAR SET OF COUPLED ORDINARY NONLINEAR DIFFERENTIAL EQUATIONS

INTRODUCTION

The mathematical modeling of certain physical phenomena can be treated by many basic methods. Certain cases may lead to systems of nonlinear differential equations which can only be investigated by approximate methods. This paper discusses a system of equations of the form

$$A\ddot{Q} = BG - C\dot{Q}^2 - D\dot{Q} - EQ,$$

where A,B,C,D and E are square coefficient matrices and $G, Q, \dot{Q}, \ddot{Q}$ and $\dot{Q}^2$ are column vectors. Here $\dot{Q}^2$ is the column vector

$$\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{q}_n^2 \end{bmatrix} .$$

A method by which numerical solutions may be effected is also discussed.

PARTICULAR SET OF EQUATIONS

Consider the system of differential equations written in matrix form as below,

$$A\ddot{Q} = BG - C\dot{Q}^2 - D\dot{Q} - EQ, \tag{1}$$

where the matrices and vectors are real and A is nonsingular.

If A is a square nonsingular matrix then it possesses an inverse, A^{-1}. Multiplying through by A^{-1} gives

$$\ddot{Q} = A^{-1}BG - A^{-1}C\dot{Q}^2 - A^{-1}D\dot{Q} - EQ \tag{2a}$$

or

$$\ddot{Q} = JG - K\dot{Q}^2 - L\dot{Q} - MQ, \tag{2b}$$

where

$$J = A^{-1}B$$

$$K = A^{-1}C$$

$$L = A^{-1}D$$

$$M = E.$$

If Q is a column vector of order n, then Q can be expressed as

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{bmatrix}, \tag{3}$$

and its time derivatives as

$$\dot{Q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{bmatrix} \tag{4}$$

and

$$\ddot{Q} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \cdot \\ \cdot \\ \cdot \\ \ddot{q}_n \end{bmatrix} . \tag{5}$$

If A, B, C, D, and E are of (n x n) order, the matrix multiplication in (2a) will yield n ordinary coupled differential equations which are nonlinear due to the $\dot{q}_i^2$ terms as shown below,

$$\ddot{q}_1 = f_1 \; (t, q_1, q_2, \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, \dot{q}_1^2, \ldots \dot{q}_n^2) \tag{6a}$$

$$\ddot{q}_2 = f_2 \; (t, q_1, q_2, \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, \dot{q}_1^2, \dot{q}_2^2, \ldots \dot{q}_n^2) \tag{6b}$$

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

$$\ddot{q}_n = f_n \; (t, q_1, q_2, \ldots q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, \dot{q}_1^2, \dot{q}_2^2, \ldots \dot{q}_n^2) . \tag{6c}$$

At the present time, a closed-form solution to these equations is not obtainable. It should be noted that for some physical problems, the coefficient matrices may possess time-dependent elements; if so, equations (6) become a set of nonautonomous, nonlinear, differential equations.

A BRIEF BACKGROUND ON THE RUNGE-KUTTA METHOD

The solution of nonlinear differential equations in closed analytic form is not usually possible. The demands of applied science make it necessary to obtain some insight into the nature of solutions subject to given boundary conditions. Graphical and numerical representation of the functions are the usual methods employed.[1]

The numerical solution of differential equations by a method of finite integration in one of its several forms is a favorite tool when performed by means of digital computers. The Runge-Kutta method is used perhaps more than any other, and will be reviewed here briefly.[2]

Consider a system of first-order simultaneous differential equations. For simplicity, choose two as given below (note that the method can be extended to n equations).

$$\frac{dy}{dt} = f(t, y, z) \qquad (7)$$

$$\frac{dz}{dt} = g(t, y, z).$$

The estimate for the function y and z at some later time (t + h) can be computed by the relations

$$y(t) = y_o + \Delta y = y_o + 1/6\ (\Delta' z + 2\Delta'' y + 2\Delta''' y + \Delta^4 y)$$

and (8)

$$z(t) = z_o + \Delta z = z_o + 1/6\ (\Delta' z + 2\Delta'' z + 2\Delta''' z + \Delta^4 z).$$

The increment values (Δ^i) are computed from:

$$\Delta' y = f(t_o, y_o, z_o)\ h \qquad (9)$$

$$\Delta' z = g(t_o, y_o, z_o)\ h$$

$$\Delta'' y = f(t_o + 1/2h,\ y_o + 1/2\ \Delta' y,\ z_o + 1/2\ \Delta' z)\ h \qquad (10)$$

$$\Delta'' z = g(t_o + 1/2h,\ y_o + 1/2\ \Delta' y,\ z_o + 1/2\ \Delta' z)\ h$$

$$\Delta'''y = f(t_o + 1/2h,\ y_o + 1/2\ \Delta''y_o,\ z_o + 1/2\ \Delta''z)\ h \tag{11}$$
$$\Delta'''z = g(t_o + 1/2h,\ y_o + 1/2\ \Delta''y,\ z_o + 1/2\ \Delta''z)\ h$$

$$\Delta^4 y = f(t_o + h,\ y_o + \Delta'''y,\ z_o + \Delta'''z)\ h \tag{12}$$
$$\Delta^4 z = g(t_o + h,\ y_o + \Delta'''y,\ z_o + \Delta'''z)\ h.$$

If the increments of y and z are computed <u>in the order given</u>, only the previously computed increments are needed in each step in the computation (i.e., the increments cannot be computed simultaneously).

This method can be applied to a system of higher-order equations without essential change. Consider the system of second-order equations

$$\frac{d^2y}{dt^2} = f(t,\ y,\ z,\ \frac{dy}{dt},\ \frac{dz}{dt}\ ) \tag{13}$$

and

$$\frac{d^2z}{dt^2} = g(t,\ y,\ z,\ \frac{dy}{dt},\ \frac{dz}{dt}\ ).$$

Introducing the new variables

$$u = \frac{dy}{dt} \text{ and } v = \frac{dz}{dt}, \tag{14}$$

the system of equations given by (13) becomes

$$\frac{du}{dt} = f(t,\ y,\ z,\ u,\ v)$$

and (15)

$$\frac{dv}{dt} = g(t,\ y,\ z,\ u,\ v).$$

This system of equations can now be numerically integrated, using the method described earlier.

Consider the system of equations given in (6) earlier. The Runge-Kutta integration method requires that the

highest-order derivative of only one dependent variable be on the left side of the equality in each equation, since the estimate of each function (see equations (9) through (12)) is computed in a stepwise fashion, using the increment values Δ^i. An attempt to numerically integrate a system of equations with more than one highest-order derivative on either side of the equality will make the Runge-Kutta algorithm unstable. The set of equations given in (6) are now of a form which is acceptable to the Runge-Kutta algorithm.

Note that the elements of the coefficient matrices may be time-dependent. If so, they can be computed at each time increment before the matrix operations take place. Most digital computers are equipped with built-in subroutines for the evaluation of matrices and determinants. These can be incorporated within the numerical integration loop as the integration is carried on.

EXAMPLE

Consider the equation of motion of a towed body moving through a nonaccelerating viscous fluid.[3,4] In body coordinates, the equations become, in matric form,

$$M_s\ddot{Q} - M_sG + H + T = 0, \tag{16}$$

where

M_s = the body structural mass matrix
Q = the displacement vector of the body
G = the gravitational vector
H = the hydrodynamic force vector
T = the towline tension vector.

The hydrodynamic force vector (from boundary layer theory[5]) can be written as

$$H = M_h\ddot{Q} + D, \tag{17}$$

where

M_h = the body hydrodynamic mass matrix
= m_{hij} (see appendix)
D = the viscous force vector.

Substituting (17) into (16) gives

$$(M_s + M_h)\ \ddot{Q} = MG - D - T. \tag{18}$$

If the viscous force vector is considered to be given in the form

$$D = C_1\dot{Q}^2 + C_2\dot{Q}, \tag{19}$$

where C_1 and C_2 are viscous coefficient (i.e., lift, drag, linear-damping coefficients or stability derivatives) matrices for the body in question, substitution into (12) gives

$$(M_s + M_h)\ \ddot{Q} = M_sG - C_1\dot{Q}^2 - C_2\dot{Q} - T, \qquad (19)$$

which is of the same form as (1) discussed previously.

For a body with six degrees of freedom, there are 6 x 2 = 12 equations which describe the body motion (i.e., acceleration and velocity) and the necessary auxiliary relations which must be found to compute the towline tension.[3,4]

Using the method described in the first section, these equations can be numerically integrated when put into the form as given by (6). Note that the elements of the hydrodynamic mass matrix may be time- or amplitude-dependent.[6,7] For this case, they can be recomputed at the end of each time increment in the numerical integration before the matr..x operations in (2) are performed.

SUMMARY

It has been noted that a particular set of nonlinear differential equations may be reduced to a form such that numerical integration of the equations may be effected using a Runge-Kutta method. One example of a physical system to which a set of equations of this form may apply has been provided to illustrate the problem. It is the intent of this paper to point out one approximate method of solution to these equations as an aid to anyone confronted by a similar problem.

APPENDIX

HYDRODYNAMIC MASS

For the case of a body moving through a fluid, the effects on body motions due to hydrodynamic inertia (or hydrodynamic mass) terms must sometimes be accounted for by the mathematical model used. For a body moving in an unsteady manner, the fluid disturbance due to the body motion extends in decreasing amplitude to infinity. From a system of reference moving with the body, the effect can be considered as localized to a finite volume of fluid surrounding the body. The body acts as if an added mass of fluid is moving with it. If the body is subjected to an acceleration, not only must the body mass be accelerated but also the added mass of fluid.[8]

Writing Newton's Second Law for the body gives

$$F = (m + m_h)\,\vec{a}\ .$$

Here m is the mass of the body and m_h is the added mass or hydrodynamic mass of the body.

The hydrodynamic mass is expected to be proportional to the fluid density, body size (i.e., its volume), and its orientation while in motion through the fluid. For general body cases with six motion degrees of freedom, there are 21 hydrodynamic inertia components.[6] These are best given in the form of a symmetric tensor. Specifically,

$$m_{h_{ij}} = -\rho \int \frac{\partial \phi i}{\partial n}\ \phi_j\ dA \qquad \begin{array}{l} i = 1, 2, 3, \ldots 6 \\ j = 1, 2, 3, \ldots 6, \end{array}$$

where ϕ_i is the normalized velocity potential caused by a unit motion from some reference position, and it can be shown that the various $\frac{\partial \phi i}{\partial n}$ depend only on body shape.[6,8]

Miller[9] has shown that the hydrodynamic mass values may be both frequency and amplitude dependent for irregular shaped bodies. For this condition, the set of differential equations used to model the body motion become nonautonomous. The techniques described in the body of the report

may be useful in obtaining approximate solutions to the types of equations used to model this problem.

The effects of hydrodynamic inertia terms on some physical systems may be significant and should be considered when writing equations of motion for such systems. If they are determined to be negligible as applied to a particular system, they should, by all means, be neglected in order to effect solutions. However, in the case of certain body motions in a viscous fluid, the equations of motion may become highly nonlinear due to the effects of damping and hydrodynamic inertia. For this case, numerical methods of solution may be required.

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